

ON THE EFFECTS OF BOHM'S POTENTIAL ON A MACROSCOPIC SYSTEM OF SELF-INTERACTING PARTICLES

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Abstract. We consider an instationary macroscopic system of self-interacting particles with an additional potential, the so called Bohm's potential. We study the existence of non-negative global solutions to the (4-th order) system of equations and allude the differences to results obtained for classical models. The problem is considered on a bounded domain up to three space dimension, subject to initial and Dirichlet boundary condition for the self-interacting potential and Neumann boundary condition for the particle density. Moreover, the initial datum is only assumed to be non-negative and to satisfy a weak integrability condition.

Key words. Higer order parabolic PDE, global solution, positivity, entropy, Bohm's potential, self-interaction, singular perturbation

AMS subject classifications.

1. Introduction. Consider a macroscopic system of self-interacting particles with *Bohm's potential*, which describe the evolution of the *normalized density* $n \geq 0$,

$$(1a) \quad n_t = \operatorname{div}(n \nabla F) \quad \text{in } \Omega, \quad n \partial_\nu F = 0 \quad \text{on } \Gamma,$$

with the *quasi fermi-level* F given by

$$(1b) \quad F = -\epsilon^2 \frac{\Delta \sqrt{n}}{\sqrt{n}} + \log n - \sigma \Phi \quad \text{in } \Omega, \quad \partial_\nu \sqrt{n} = 0 \quad \text{on } \Gamma,$$

and the *potential* Φ due to self-interaction in a particle system,

$$(1c) \quad -\Delta \Phi = n \quad \text{in } \Omega, \quad \Phi = 0 \quad \text{on } \Gamma,$$

supplemented with the initial condition $n(0, \cdot) = n_0 \geq 0$, where ν is the outer normal to the convex, bounded domain $\Omega \subset \mathbb{R}^d$, $d \leq 3$ with Lipschitz boundary Γ , $\epsilon > 0$ is the scaled Planck constant and $\sigma \in \mathbb{R}$ is the mass of the system of self-interacting particles. In this case, positive mass $\sigma > 0$ would indicate the presence of self-attraction, while negative mass $\sigma < 0$ self-repulsion. Observe that the presence of the Bohm potential $\sqrt{n}^{-1} \Delta \sqrt{n}$, which is a non-local second-order term, leads to a fourth-order evolution equation for the normalized density n , given by

$$(2a) \quad n_t = \operatorname{div} \left(n \nabla \left(-\epsilon^2 \frac{\Delta \sqrt{n}}{\sqrt{n}} + \log n - \sigma \Phi \right) \right) \quad \text{in } \Omega,$$

with the natural boundary conditions

$$(2b) \quad n \partial_\nu \left(-\epsilon^2 \frac{\Delta \sqrt{n}}{\sqrt{n}} + \log n - \sigma \Phi \right) = 0 \quad \text{on } \Gamma, \quad \text{and} \quad \partial_\nu \sqrt{n} = 0 \quad \text{on } \Gamma.$$

Clearly, the boundary conditions for n imply $\int_\Omega n \, dx = \int_\Omega n_0 \, dx$. Therefore, it is sufficient to ensure that $\int_\Omega n_0 \, dx = 1$ for n to be kept normalized.

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Such kind of systems occur typically in the theory of transport equations for semiconductors as a macroscopic limit (quantum drift-diffusion equations) of its microscopic counterpart (Wigner-Boltzman-Poisson system) (c.f. [13, 16] and references therein). In this setting, n describes the electron density and therefore $\sigma < 0$, since electrons are negatively charged. In the case of no interactions, $F = -\sqrt{n}^{-1}\Delta\sqrt{n}$, also called the Derrida-Lebowitz-Speer-Spohn (DLSS) equations, the authors in [10] and [15] successfully proved the existence of solutions and additionally showed the rate of convergence of the solutions to the unique stationary solution. Many of the ideas in this paper follow as a result of these two papers.

By passing to the limit $\epsilon \rightarrow 0$, we formally recover either the classical drift-diffusion equations ($\sigma < 0$) or the standard model for a system of self-gravitating particles ($\sigma > 0$) [3]. This system is well known to have global solutions for $\sigma \in (-\infty, 8\pi)$ and blowup phenomena for large mass $\sigma > 8\pi$ in $d = 2$. Another model of the limit system comes from statistical mechanics for vortex points as the mean field limit of the canonical Gibbs measure associated to a N -vortex system in a bounded two-dimensional domain [5, 6]. Here, the authors established the concentration phenomena for the weak limits of the Gibbs measures, when $N \rightarrow \infty$, to the solution of the limit system ($\epsilon = 0$) in the case $\sigma \in (-\infty, 8\pi)$. Similar systems that manifest blowup phenomena arises in the theory of combustion [2] and chemotaxis equations [24].

The purpose of this paper, among other things, is to show that the presence of the Bohm potential ($\epsilon > 0$) leads to a regularization of the limit problem ($\epsilon = 0$). More precisely, it is shown that (1) possesses at least one global weak solution for any $\sigma \in \mathbb{R}$, i.e., the presence of the Bohm potential prevents any blowup phenomena. A short discussion on stationary solutions to (1) for $\sigma \in \mathbb{R}$ can be found in [19]. There it is also shown that stationary solutions with $\epsilon > 0$, $d = 2$ and $\sigma \in (-\infty, 8\pi)$, converge weakly to stationary solutions of the limit problem ($\epsilon = 0$).

For notational convenience, we set

$$V := \{u \in H^2(\Omega) \mid \partial_\nu u = 0 \text{ on } \Gamma\}, \quad V_0 := H^2(\Omega) \cap H_0^1(\Omega),$$

and denote the space of probability measures with finite entropy \mathcal{E} by

$$\mathcal{P} := \left\{ u \in L^1(\Omega) \mid u \geq 0, \int_\Omega u \, dx = 1, \mathcal{E}(u) < +\infty \right\},$$

where \mathcal{E} is simply the (negative) *physical entropy* given by

$$\mathcal{E}(u) := \int_\Omega (u(\log u - 1) + 1) \, dx.$$

The main purpose of this paper is to show the following result.

THEOREM 1. *For any finite $T > 0$ and $n_0 \in \mathcal{P}$, there exists at least one solution (n, F, Φ) to system (1), with*

$$\begin{aligned} n &\in W^{1,1}(0, T; V^*), \quad \sqrt{n} \in L^2(0, T; V), \\ F\sqrt{n} &\in L^2(0, T; L^2(\Omega)) \quad \text{and} \quad \Phi \in L^1(0, T; V_0), \end{aligned}$$

and additionally $n \geq 0$ a.e. in $(0, T) \times \Omega$.

The proof of Theorem 1 relies on the fact that the physical entropy \mathcal{E} provides a controlled growth estimate for (2). Indeed, by formally multiplying (2a) with $\log(n)$, integrating over Ω and integrating by parts, we obtain

$$\frac{d}{dt} \mathcal{E}(n) + c_0 \int_\Omega |\nabla^2 \sqrt{n}|^2 \, dx \leq c_1,$$

for some constant $c_0, c_1 > 0$, independent of n . By using interpolation inequalities on derivatives [1], i.e., there exist constants $\delta > 0$ and $c(\delta) > 0$ such that

$$\|\nabla u\|_2^2 \leq \delta \|\nabla^2 u\|_2^2 + c(\delta) \|u\|_2^2,$$

along with the logarithmic-Sobolev inequality [23]

$$\int_{\Omega} u^2 \log \left(\frac{u^2}{\|u\|_2^2} \right) dx \leq c_L \|\nabla u\|_2^2,$$

where $c_L > 0$ only depends on Ω and d , and the fact that $\|n\|_1 = 1$, we further obtain

$$\frac{d}{dt} \mathcal{E}(n) + c_2 \mathcal{E}(n) \leq c_3,$$

for some constants $c_2, c_3 > 0$. An application of Grownwall's lemma on this estimate provides the global boundedness in time t of $\mathcal{E}(n)$, and consequently shows (formally) the absence of a blowup phenomena in the space \mathcal{P} .

2. Preliminary results. In this section we recall several standard results regarding unique solvability and regularity of solutions for elliptic equations. We begin by recalling a well known interpolation theorem for Sobolev spaces, namely the Gagliardo-Nirenberg-Sobolev inequalities [26].

PROPOSITION 2. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain, $m \in \mathbb{N}$ and $1 \leq p, q, r \leq \infty$. Then there exists a constant $c > 0$, such that*

$$\|D^\alpha u\|_r \leq c \|u\|_{m,p}^\theta \|u\|_q^{1-\theta} \quad \text{for any } u \in W^{m,p}(\Omega) \cap L^q(\Omega),$$

provided that $0 \leq |\alpha| \leq m-1$, $\theta = |\alpha|/m$ and $|\alpha| - d/r = \theta(m - d/p) - (1-\theta)d/q$. If $m - |\alpha| - d/p \neq \mathbb{N}_0$, then the values $|\alpha|/m \leq \theta \leq 1$ are allowed.

We also recall a regularity result for linear elliptic problems on convex, bounded domains due to [12].

PROPOSITION 3. *Let $\Omega \subset \mathbb{R}^d$ be a convex, bounded domain and $f \in L^2(\Omega)$. Then the homogeneous Dirichlet problem*

$$-\Delta u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \Gamma,$$

possesses a unique weak solution $u \in V_0$, which satisfies the estimate

$$\|u\|_{2,2} \leq c \|\Delta u\|_2,$$

for some constant $c > 0$, depending only on the diameter of Ω .

The next result we want to recall is a regularity result for linear elliptic equations with homogeneous Neumann boundary conditions and source terms from the Orlicz space $L \log L(\Omega)$ (c.f. [4, 22]). For the sake of completeness, we include a simple proof for this result in Appendix A.

PROPOSITION 4. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain, $f \in L \log L(\Omega)$ and $a \in L^\infty(\Omega)$ with $a \geq \alpha > 0$ a.e. in Ω . Then the elliptic equation*

$$(3) \quad -\operatorname{div}(a \nabla u) = f \quad \text{in } \Omega, \quad \partial_\nu u = 0 \quad \text{on } \Gamma,$$

has a unique weak solution $u \in W_\beta$, with $(1 + |u|)^{1/2} \in H^1(\Omega)$, where

$$W_\beta := \left\{ u \in W^{1, \frac{d}{d-1}}(\Omega) \mid \int_{\Omega} u \, dx = \beta \right\} \quad \text{with} \quad \|u\|_{W_\beta} := \|\nabla u\|_{\frac{d}{d-1}},$$

and $\beta \in \mathbb{R}$ is some given constant.

Another result we will need is a result regarding nonlinear elliptic equations with natural gradient growth [19] (c.f. [8, 9]).

PROPOSITION 5. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and $f \in L^p(\Omega)$, $p > d/2$. Then the elliptic equation*

$$-\frac{\epsilon^2}{2} \left(\Delta y + \frac{1}{2} |\nabla y|^2 \right) + y = f \quad \text{in } \Omega, \quad \partial_\nu y = 0 \quad \text{on } \Gamma,$$

has a weak solution $y \in H^1(\Omega) \cap L^\infty(\Omega)$. Furthermore, $e^{y/2} \in H^1(\Omega) \cap L^\infty(\Omega)$.

3. Existence of solutions. The strategy for proving the existence of solutions involves first showing existence of solutions for a time-discrete problem with time step $\tau > 0$ with the help of Leray-Schauder's fixed point theorem. By establishing uniform bounds on the sequence of solutions $\{n^{(\tau)}\}$ with respect to τ then allows to extract a subsequence, which converges to a solution of (1) when passing to the limit $\tau \rightarrow 0$.

3.1. Existence of a time-discrete solution. Let $\tau > 0$ an arbitrary but fixed time step and $w \in \mathcal{P}$ be a given function. The task at hand is to find a weak solution $(F, \sqrt{n}, \Phi) \in W_\beta \times V \times V_0$, for some $\beta \in \mathbb{R}$, to the semi-discrete equation

$$(4a) \quad -\operatorname{div}(n \nabla F) = (w - n)/\tau \quad \text{in } \Omega, \quad n \partial_\nu F = 0 \quad \text{on } \Gamma,$$

$$(4b) \quad -\epsilon^2 \frac{\Delta \sqrt{n}}{\sqrt{n}} + \log n = \sigma \Phi + F \quad \text{in } \Omega, \quad \partial_\nu \sqrt{n} = 0 \quad \text{on } \Gamma,$$

$$(4c) \quad -\Delta \Phi = n \quad \text{in } \Omega, \quad \Phi = 0 \quad \text{on } \Gamma,$$

where F , n and Φ satisfy (1a), (1b) and (1c) respectively.

We mention some of the problems encountered in solving this problem. Observe that, in order to solve (4a) for F , we have to make sure that $n \in L^\infty(\Omega)$ is uniformly bounded below away from zero. Otherwise, we will have to rely on degenerate elliptic estimates, which will require some kind of regularity on both n and n^{-1} .

To circumvent both these problems simultaneously, we introduce a transformation for n , given by $n = e^y / \|e^y\|_1$. Clearly, if $y \in L^\infty(\Omega)$, then n satisfies the assumptions in Proposition 4. Moreover, the normality of n is satisfied trivially. Inserting this into (4b), leads to a nonlinear elliptic equation with natural gradient growth,

$$(5) \quad -\frac{\epsilon^2}{2} \left(\Delta y + \frac{1}{2} |\nabla y|^2 \right) + y = \sigma \Phi + F + \log \|e^y\|_1 \quad \text{in } \Omega, \quad \partial_\nu y = 0 \quad \text{on } \Gamma.$$

We will then use (5) to construct an auxiliary problem, similar to (4), and apply the Leray-Schauder theorem on this problem. Under the appropriate regularity, we may then recover solutions of the original problem (4).

LEMMA 6. *Let $d \leq 3$ and $w \in \mathcal{P}$. Then there exists at least one weak solution*

$$(F, \sqrt{n}, \Phi) \in W_\beta \times V \times V_0,$$

to (4) with $\beta = -|\Omega| \log \|e^y\|_1$ and $n \in \mathcal{P}$.

Proof. Let $w \in \mathcal{P}$. For arbitrarily given $v \in \mathcal{C}(\overline{\Omega})$ and $\lambda \in [0, 1]$, we consider the auxiliary problem to find $(y, F, \Phi) \in V \times W_\beta \times V_0$, with $\beta = -\lambda |\Omega| \log \|e^v\|_1$:

$$(6a) \quad -\operatorname{div}((e^v / \|e^v\|_1) \nabla F) = \lambda(w - (e^v / \|e^v\|_1))/\tau \quad \text{in } \Omega, \quad \partial_\nu F = 0 \quad \text{on } \Gamma,$$

$$(6b) \quad -\frac{\epsilon^2}{2} \left(\Delta y + \frac{1}{2} |\nabla y|^2 \right) + y = \sigma \Phi + F + \lambda \log \|e^v\|_1 \quad \text{in } \Omega, \quad \partial_\nu y = 0 \quad \text{on } \Gamma,$$

$$(6c) \quad -\Delta \Phi = \lambda(e^v / \|e^v\|_1) \quad \text{in } \Omega, \quad \Phi = 0 \quad \text{on } \Gamma,$$

As mentioned above, we used the variable transformation $n = e^y / \|e^y\|_1$. Note also that for the case $\lambda = 1$, system (6) is equivalent to the initial system (1).

We begin by showing that this system yields a weak solution $y \in V$ for any $v \in \mathcal{C}(\overline{\Omega})$ and $\lambda \in [0, 1]$. Indeed, since $e^v / \|e^v\|_1 \in \mathcal{C}(\overline{\Omega})$ and is uniformly bounded from below by a positive constant, we obtain a unique solution $F \in W_\beta$ for (6a) due to Proposition 4. From Proposition 3, we obtain a unique solution $\Phi \in V_0$ for (6c). Now, by inserting (F, Φ) into (6b), we obtain a solution $y \in H^1(\Omega) \cap L^\infty(\Omega)$, due to Proposition 5. Note also that $n \in L^\infty(\Omega)$ with $n \geq e^{\inf y} / \|e^y\|_1 =: \underline{n} > 0$ a.e. in Ω and $\int_\Omega n \, dx = 1$. Furthermore, by rewriting (6b) in terms of $\rho := \sqrt{n}$, we have

$$-\epsilon^2 \Delta \rho = \rho (\sigma \Phi + F - \log \rho) \in L^2(\Omega),$$

which implies $\Delta \rho \in L^2(\Omega)$, since $\rho \in L^\infty(\Omega)$. Due to the convexity of Ω and the homogeneous Neumann boundary condition for ρ , we have the second order inequality $\|\nabla^2 \rho\|_2 \leq \|\Delta \rho\|_2$, where $\nabla^2 \rho$ denotes the Hessian of ρ [12]. Therefore, $\rho \in H^2(\Omega)$ and consequently $y \in V$, since

$$(7) \quad \partial_{ij} y = 2 \partial_i (\rho^{-1} \partial_j \rho) = 2(\rho \partial_{ij} \rho - \partial_i \rho \partial_j \rho) / \rho^2 \in L^2(\Omega),$$

Due to the boundary condition for y , we further have $\partial_\nu \rho = 0$ a.e. on Γ , i.e., $\rho \in V$.

Now consider the operator $H: \mathcal{C}(\overline{\Omega}) \times [0, 1] \rightarrow \mathcal{C}(\overline{\Omega})$; $(v, \lambda) \mapsto y$. This operator is continuous and compact due to the continuity of the solution operators corresponding to (6a)–(6c) respectively and the compact embedding $H^2(\Omega) \hookrightarrow \mathcal{C}(\overline{\Omega})$. It is also easy to see that $H(v, 0) = 0$ for all $v \in \mathcal{C}(\overline{\Omega})$. We see this by simply testing the variational formulation of (6b) with $\varphi = \text{sign}(y)(e^{|y|} - 1)$, which yields

$$\epsilon^2 \int_\Omega |\nabla e^{|y|/2}|^2 \, dx + \frac{\epsilon^2}{4} \int_\Omega |\nabla y|^2 \, dx + \int_\Omega |y|^2 \, dx \leq 0,$$

where we used the fact that $s(e^s - 1) \geq s^2$ for $s \geq 0$. Therefore, $y = 0$ a.e. in Ω and consequently $y = 0$ in $\overline{\Omega}$, since y is continuous.

Let $(y, \lambda) \in \mathcal{C}(\overline{\Omega}) \times [0, 1]$ such that $H(y, \lambda) = y$. We now show that y is uniformly bounded (in λ) by some constant $M > 0$ in $\mathcal{C}(\overline{\Omega})$. Observe that for $\sqrt{n} \in V$, the auxiliary system (6) is equivalent to the equations

$$(8a) \quad \frac{\lambda}{\tau} (n - w) = \text{div} \left(n \nabla \left(-\epsilon^2 \frac{\Delta \sqrt{n}}{\sqrt{n}} + \log n - \sigma \Phi \right) \right)$$

$$(8b) \quad -\Delta \Phi = \lambda n,$$

where we also used the fact that $\log \|e^y\|_1$ is constant. Since $\phi(s) = s(\log s - 1) + 1$ is convex, $\phi(s_1) - \phi(s_2) \leq \phi'(s_1)(s_1 - s_2)$ for all $s_1, s_2 \geq 0$. Therefore,

$$\begin{aligned} \frac{\lambda}{\tau} (\mathcal{E}(n) - \mathcal{E}(w)) &= \frac{\lambda}{\tau} \int_\Omega (\phi(n) - \phi(w)) \, dx \leq \frac{\lambda}{\tau} \int_\Omega (n - w) \log n \, dx \\ &= - \int_\Omega n \nabla \left(-\epsilon^2 \frac{\Delta \sqrt{n}}{\sqrt{n}} + \log n - \sigma \Phi \right) \cdot \nabla \log n \, dx \\ &= \epsilon^2 \int_\Omega n \nabla \left(\frac{\Delta \sqrt{n}}{\sqrt{n}} \right) \cdot \nabla \log n \, dx - \int_\Omega n |\nabla \log n|^2 \, dx + \sigma \int_\Omega \nabla \Phi \cdot \nabla n \, dx \\ &= \epsilon^2 J_1 - J_2 + \sigma J_3. \end{aligned}$$

For the following computations we set $\rho = \sqrt{n}$. For J_1 , we have the following

$$\begin{aligned} J_1 &= 2 \int_{\Omega} \left(\nabla(\rho \Delta \rho) - 2\Delta \rho \nabla \rho \right) \cdot \frac{\nabla \rho}{\rho} dx \\ &= -2 \int_{\Omega} |\Delta \rho|^2 + \Delta \rho \frac{|\nabla \rho|^2}{\rho} dx \leq -2 \int_{\Omega} \rho^2 \left[\left| \frac{\nabla^2 \rho}{\rho} \right|^2 + \frac{\Delta \rho}{\rho} \left| \frac{\nabla \rho}{\rho} \right|^2 \right] dx, \end{aligned}$$

where, in the last inequality, we used the fact that $\|\nabla^2 \rho\|_2 \leq \|\Delta \rho\|_2$, due to the results in [12]. As for J_2 , we have the *Fisher information*

$$J_2 = \int_{\Omega} n |\nabla \log n|^2 dx = 4 \int_{\Omega} \rho^2 \left| \frac{\nabla \rho}{\rho} \right|^2 dx.$$

Since Φ satisfies (8b), we have

$$J_3 = - \int_{\Omega} (\Delta \Phi) n dx + \int_{\Gamma} n \partial_{\nu} \Phi ds = \lambda \int_{\Omega} n^2 dx + \int_{\Gamma} n \partial_{\nu} \Phi ds.$$

Due to the regularity of Φ and the boundary condition $\Phi = 0$ on Γ , we have $\partial_{\nu} \Phi \leq 0$. Therefore, $J_3 \leq \int_{\Omega} n^2 dx$ for any $\lambda \in [0, 1]$. Altogether, we have

$$\frac{\lambda}{\tau} (\mathcal{E}(n) - \mathcal{E}(w)) \leq -2 \int_{\Omega} \rho^2 \left[\epsilon^2 \left| \frac{\nabla^2 \rho}{\rho} \right|^2 + \epsilon^2 \frac{\Delta \rho}{\rho} \left| \frac{\nabla \rho}{\rho} \right|^2 + 2 \left| \frac{\nabla \rho}{\rho} \right|^2 - \frac{\sigma}{2} n \right] dx.$$

From Appendix B, we obtain the estimate

$$\int_{\Omega} n^2 dx \leq \delta_1 \int_{\Omega} n \left| \frac{\nabla \rho}{\rho} \right|^4 dx + \delta_2 \int_{\Omega} n \left| \frac{\nabla \rho}{\rho} \right|^2 dx + c(\delta_1, \delta_2, \alpha),$$

for any $\delta_1, \delta_2 > 0$ and some $\alpha \in (0, 1)$ for $d = 2$ or $\alpha \in (1/3, 1)$ for $d = 3$. By choosing $\delta_2 = 4/|\sigma|$ and rescaling $\delta_1 = 2\epsilon^2 \delta / |\sigma|$ for some $\delta > 0$, we further obtain

$$(9) \quad \frac{\lambda}{\tau} (\mathcal{E}(n) - \mathcal{E}(w)) + 2\epsilon^2 \int_{\Omega} \rho^2 \left[\left| \frac{\nabla^2 \rho}{\rho} \right|^2 + \frac{\Delta \rho}{\rho} \left| \frac{\nabla \rho}{\rho} \right|^2 - \delta \left| \frac{\nabla \rho}{\rho} \right|^4 \right] dx \leq c(\delta).$$

In Appendix C, we show that the second term on the left is bounded from below by a multiple of $\|\nabla^2 \rho\|_2^2$, i.e., we have for some $\gamma > 0$ (explicitly given below) the estimate

$$(10) \quad \frac{\lambda}{\tau} (\mathcal{E}(n) - \mathcal{E}(w)) + 2\gamma \epsilon^2 \int_{\Omega} |\nabla^2 \rho|^2 dx \leq c(\delta) \quad \text{for any } \lambda \in [0, 1],$$

which gives the uniform bounds

$$(11) \quad \mathcal{E}(n) \leq \mathcal{E}(w) + c(\delta)\tau \quad \text{and} \quad \|\nabla^2 \rho\|_2^2 \leq \frac{1}{2\gamma \epsilon^2 \tau} (\mathcal{E}(w) + c(\delta)\tau),$$

thus implying the boundedness of $\nabla^2 \rho$ in $L^2(\Omega)$ independent of $\lambda \in [0, 1]$, and therefore the uniform boundedness of $\rho \in V$. As a consequence, we obtain an upper bound for y , i.e. there exists some constant $M_u > 0$, such that $y < M_u$ a.e. in Ω .

To show that y is uniformly bounded from below, we have to show a uniform lower bound for ρ away from zero. This is a result of the Harnack inequality [20, 25].

Firstly, note that $\log \rho - \sigma \Phi - F \in L^p(\Omega)$, $p > d/2$. Then by Harnack's inequality, a weak solution $\rho \in H_{loc}^1(\Omega)$ of the equation

$$-\epsilon^2 \Delta \rho + \mu \rho = 0 \quad \text{in } \Omega, \quad \text{with } \mu \in L^p(\Omega), \quad p > d/2,$$

in a cube $Q = Q(3r) \subset \Omega$ with $0 \leq \rho \leq M_u$ in Q , satisfies

$$\max_{Q(r)} \rho(x) \leq c_h \min_{Q(r)} \rho(x),$$

for some constant $c_h = c_h(\lambda) > 0$ independent of ρ .

Now it is an easy exercise to verify that $\rho(x)$ for some $x \in \Omega$ would lead to $\rho \equiv 0$ in Ω , which clearly contradicts $\int_{\Omega} \rho^2 dx = 1$. Therefore, $\rho \geq \underline{\rho} > 0$ a.e. in Ω uniformly (in λ). Hence, there exists some constant $M_l > 0$ such that $y > -M_l$ a.e. in Ω . Choosing $M = \max\{M_l, M_u\}$ gives the estimate $\|y\|_{\infty} < M$. Furthermore, we see from (7) that $\nabla^2 y \in L^2(\Omega)$ is uniformly bounded. This implies that y is uniformly bounded in $H^2(\Omega)$, and consequently in $\mathcal{C}(\overline{\Omega})$ with the same constant M , due to the Sobolev embedding $H^2(\Omega) \hookrightarrow \mathcal{C}(\overline{\Omega})$. We finally conclude the proof by applying the Leray-Schauder fixed point theorem [11]. \square

REMARK 7. The value $\beta \in \mathbb{R}$ may be thought of as a Lagrange multiplier for the constraint on the density $\int_{\Omega} n dx = 1$. Indeed, the solution $\sqrt{n} \in \mathcal{P}$ of (4b) may be characterized as the unique minimizer of the functional

$$\mathcal{F}(n) := \epsilon^2 \int_{\Omega} |\nabla \sqrt{n}|^2 dx + \mathcal{E}(n) - \int_{\Omega} n (\sigma \Phi + F) dx,$$

on the space $\{n \in \mathcal{P} \mid \sqrt{n} \in V\}$ (c.f. [7]).

3.2. A priori estimates. Let $T > 0$ be a fixed arbitrary terminal time. For every $\tau > 0$ we define the step function $n^{(\tau)}: [0, T] \rightarrow L^1(\Omega)$ recursively as follows. Let $n(0) = n_0$ and for given $k \in \mathbb{N}$, let $(F_k, \sqrt{n_k}, \Phi_k) \in W_{\beta} \times V \times V_0$ be the non-negative solution of (4) with $w = n_{k-1}$. Now define

$$F^{(\tau)}(t) := F_k, \quad n^{(\tau)}(t) := n_k, \quad \Phi^{(\tau)}(t) := \Phi_k \quad \text{for } (k-1)\tau < t \leq k\tau.$$

Then $(F^{(\tau)}, n^{(\tau)}, \Phi^{(\tau)})$ satisfies

$$(12a) \quad (n^{(\tau)} - \zeta_{\tau} n^{(\tau)})/\tau = \operatorname{div}(n^{(\tau)} \nabla F^{(\tau)}) \quad \text{in } \Omega,$$

$$(12b) \quad F^{(\tau)} = -\epsilon^2 \frac{\Delta \sqrt{n^{(\tau)}}}{\sqrt{n^{(\tau)}}} + \log n^{(\tau)} - \sigma \Phi^{(\tau)} \quad \text{in } \Omega,$$

$$(12c) \quad -\Delta \Phi^{(\tau)} = n^{(\tau)} \quad \text{in } \Omega,$$

together with their respective boundary conditions. Here, ζ_{τ} denotes the shift operator $(\zeta_{\tau} n^{(\tau)})(t) = n^{(\tau)}(t - \tau)$ for $t \in [\tau, T]$. As a consequence of Lemma 6, we obtain the following uniform estimate for the sequence of step functions $\{n^{(\tau)}\}$.

LEMMA 8. There exists a $\delta_0 > 0$ such that for any $\delta \in [0, \delta_0]$, the sequence of step functions $\{n^{(\tau)}\}$ satisfies

$$\frac{1}{\tau} \|n^{(\tau)} - \zeta_{\tau} n^{(\tau)}\|_{L^{1+\delta}(\tau, T; V^*)} + \|n^{(\tau)}\|_{L^{1+\delta}(0, T; H^2(\Omega))} \leq c$$

for some constant $c > 0$ independent of τ .

Proof. For notational convenience, we set as before $\rho^{(\tau)} = \sqrt{n^{(\tau)}}$. We begin by establishing a uniform (in τ) bound for the translations. To do so, we multiply (12a) with an arbitrary function $\varphi \in V$, integrate over Ω and integrating by parts to obtain

$$(13) \quad \begin{aligned} \frac{1}{\tau} \int_{\Omega} (n^{(\tau)} - \zeta_{\tau} n^{(\tau)}) \varphi \, dx &= \int_{\Omega} F^{(\tau)} (\nabla n^{(\tau)} \cdot \nabla \varphi + n^{(\tau)} \Delta \varphi) \, dx \\ &\leq \left(\|F^{(\tau)} \nabla n^{(\tau)}\|_{\frac{4}{3}} + \|F^{(\tau)} n^{(\tau)}\|_2 \right) \|\varphi\|_V, \end{aligned}$$

where we used the embedding $H^2(\Omega) \hookrightarrow W^{1,4}(\Omega)$. This implies the estimate

$$\frac{1}{\tau} \|n^{(\tau)} - \zeta_{\tau} n^{(\tau)}\|_{L^{1+\delta}(\tau, T; V^*)}^{1+\delta} \leq c_1 \int_0^T \|F^{(\tau)} \nabla n^{(\tau)}\|_{\frac{4}{3}}^{1+\delta} + \|F^{(\tau)}(t) n^{(\tau)}(t)\|_2^{1+\delta} \, dt,$$

for some constant $c_1 > 0$. Since there are constants $c_i > 0$, such that

$$\|F \nabla n\|_{\frac{4}{3}}^{1+\delta} \leq c_2 \|F \rho\|_2^2 + c_3 \|\nabla \rho\|_4^{2\eta(\delta)} \quad \text{and} \quad \|F n\|_2 \leq c_4 \|F \rho\|_2^2 + c_5 \|\rho\|_{\infty}^{2\eta(\delta)},$$

with $\eta(\delta) = (1 + \delta)/(1 - \delta) > 1$, we further obtain

$$(14) \quad \frac{1}{\tau} \|n^{(\tau)} - \zeta_{\tau} n^{(\tau)}\|_{L^{1+\delta}(\tau, T; V^*)}^{1+\delta} \leq c_6 \int_0^T \|\rho^{(\tau)}\|_{\infty}^{2\eta(\delta)} + \|\nabla \rho^{(\tau)}\|_4^{2\eta(\delta)} + \|F^{(\tau)} \rho^{(\tau)}\|_2^2 \, dt.$$

Therefore, the required estimate follows from the uniform boundedness of the right hand side of the above equation in τ .

We now make several observations from the results obtained in Section 3.1. As a direct consequence of the conservation of mass, we obtain the uniform bound

$$\|\rho^{(\tau)}\|_{L^{\infty}(0, T; L^2(\Omega))} = \|n^{(\tau)}\|_{L^{\infty}(0, T; L^1(\Omega))} = \int_{\Omega} n_0 \, dx = 1.$$

Furthermore, from (10), we deduce the uniform bound

$$(15) \quad \|\nabla^2 \rho^{(\tau)}\|_{L^2(0, T; L^2(\Omega))} \leq c_0,$$

with a constant $c_0 > 0$ depending only on d , Ω , T , and n_0 , and consequently, the uniform boundedness of the sequence $\{\rho^{(\tau)}\} \subset L^2(0, T; V)$. In order to establish uniform boundedness of the first two terms in (14), we make use of Proposition 2. For the first term, we have

$$(16) \quad \int_0^T \|\rho\|_{\infty}^{2\eta(\delta)} \, dt \leq c_7 \|\rho\|_{L^{\infty}(0, T; L^2(\Omega))}^{2(1-\theta)\eta(\delta)} \int_0^T \|\rho\|_{2,2}^{2\theta\eta(\delta)} \, dt,$$

with $\theta = d/4 \in (0, 1)$ for $d \leq 3$, which leads to uniform boundedness if we choose $\eta(\delta) \leq 4/d$. This is equivalent to the requirement $\delta \leq (4 - d)/(4 + d) =: \delta_1$. Similarly, we have for the second term

$$(17) \quad \int_0^T \|\nabla \rho\|_4^{2\eta(\delta)} \, dt \leq c_8 \|\rho\|_{L^{\infty}(0, T; L^2(\Omega))}^{2(1-\theta)\eta(\delta)} \int_0^T \|\rho\|_{2,2}^{2\theta\eta(\delta)} \, dt,$$

with $\theta = (4 + d)/8 \in (0, 1)$ for $d \leq 3$. Here, we may choose $\eta(\delta) \leq 8/(4 + d)$, which is equivalent to choosing $\delta \leq (4 - d)/(12 + d) =: \delta_2$. Therefore, we may choose

$\delta_0 = \min\{\delta_1, \delta_2\}$. The uniform boundedness of the last term in (14) may be seen as follows. From (12b), we obtain the following estimate

$$\|F\rho\|_2 \leq \epsilon^2 \|\nabla^2 \rho\|_2 + 2\|\rho \log \rho\|_2 + |\sigma| \|\rho \Phi\|_2 \leq \epsilon^2 \|\rho\|_{2,2} + (2 + c_9 |\sigma|) \|\rho\|_4^2,$$

where we used the fact that $s \log(s) \leq s^2$ for $s > 0$, the a priori estimate provided by Proposition 3, and the Sobolev embedding $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$. Now, from Proposition 2, we deduce the existence of constants $c_{10}, c_{11} > 0$, such that

$$(18) \quad \int_0^T \|F\rho\|_2^2 dt \leq c_{10} \int_0^T \|\rho\|_{2,2}^2 dt + c_{11} \|\rho\|_{L^\infty(0,T;L^2(\Omega))}^{4(1-\theta)} \int_0^T \|\rho\|_{2,2}^{4\theta} dt,$$

with $\theta = d/8 \in (0, 1)$, which shows the uniform bound for $\{F^{(\tau)}\rho^{(\tau)}\}$. Therefore, by collecting the estimates obtained above, we conclude the first assertion.

To show that $\{n^{(\tau)}\}$ is uniformly bounded in $L^{1+\delta}(0, T; H^2(\Omega))$, we simply use the fact that $\nabla^2 n = 2(|\nabla \rho|^2 + \rho \nabla^2 \rho)$ and Proposition 2 again, to obtain

$$\int_0^T \|\nabla^2 n\|_2^{1+\delta} dt \leq c_{12} \int_0^T \|\nabla \rho\|_4^{2(1+\delta)} + \|\rho\|_\infty^{2\eta(\delta)} + \|\nabla^2 \rho\|_2^2 dt.$$

Since $(1 + \delta) \leq \eta(\delta)$, we have the uniform boundedness of the right hand side due to the estimates (15), (16) and (17). Similarly, we can show the uniform bound for $\{n^{(\tau)}\}$ in $L^{1+\delta}(0, T; L^2(\Omega))$, which leads to the estimate asserted for $\{n^{(\tau)}\}$. \square

3.3. Passing to the limit $\tau \rightarrow 0$. We begin this section by recalling a generalized version of Aubin's lemma found in [21].

PROPOSITION 9. *Let X, B, Y be Banach spaces and $X \hookrightarrow B \hookrightarrow Y$ with compact embedding $X \hookrightarrow B$, and $F \subset L^q(0, T; B) \cap L^1(0, T; X)$, $q \in (1, \infty]$, be bounded with*

$$\|u - \zeta_\tau u\|_{L^1(\tau, T; Y)} \rightarrow 0 \text{ as } \tau \rightarrow 0, \text{ uniformly for } u \in F,$$

then F is relatively compact in $L^p(0, T; B)$ for all $p < q$.

A simple consequence of Proposition 9 is the following result.

LEMMA 10. *There exists a nonnegative function $n \in W^{1,1}(0, T; V^*)$ such that the following convergences hold true for some subsequence of $\{n^{(\tau)}\}$:*

$$\begin{aligned} (n^{(\tau)} - \zeta_\tau n^{(\tau)})/\tau &\rightharpoonup \partial_t n \text{ in } L^{1+\delta}(0, T; V^*), \quad n^{(\tau)} \rightharpoonup n \text{ in } L^{1+\delta}(0, T; H^2(\Omega)), \\ n^{(\tau)} &\rightarrow n \quad \text{and} \quad \nabla n^{(\tau)} \rightarrow \nabla n \text{ a.e. in } (0, T) \times \Omega, \end{aligned}$$

for $\tau \rightarrow 0$, where \rightharpoonup denotes the weak convergences in their respective spaces.

Proof. Since the Lebesgue-Bochner spaces $L^{1+\delta}(0, T; V^*)$ and $L^{1+\delta}(0, T; H^2(\Omega))$, with $\delta > 0$ are reflexive, they are weakly sequentially compact. Due to Lemma 8, we obtain a subsequence of $\{n^{(\tau)}\}$ (not relabeled) and some $n \in L^{1+\delta}(0, T; H^2(\Omega))$ such that the second convergence holds true. Furthermore, Proposition 9 provides yet another subsequence of $\{n^{(\tau)}\}$ such that

$$n^{(\tau)} \rightarrow n \text{ in } L^{1+\delta}(0, T; W^{1,4}(\Omega)),$$

due to the compact embedding $H^2(\Omega) \hookrightarrow W^{1,4}(\Omega)$. In particular, we may extract a subsequence such that $n^{(\tau)} \rightarrow n$ and $\nabla n^{(\tau)} \rightarrow \nabla n$ almost everywhere. Moreover, since $n^{(\tau)} \geq 0$ uniformly almost everywhere for all τ , the limit function also satisfies $n \geq 0$ almost everywhere.

In a similar way, we obtain some $\chi \in L^{1+\delta}(0, T; V^*)$ such that

$$(n^{(\tau)} - \zeta_\tau n^{(\tau)})/\tau \rightharpoonup \chi \text{ in } L^{1+\delta}(0, T; V^*)$$

for some subsequence. In order to identify this limit with the time derivative of n , we multiply n with arbitrary functions $v \in V$ and $\varphi \in \mathcal{C}_0^\infty(0, T)$, integrate over time and space, and integrating by parts w.r.t. time, we obtain

$$\begin{aligned} \int_0^T \langle n, v \rangle_V \varphi_t dt &= \lim_{\tau \rightarrow 0} \int_0^{T-\tau} \langle n^{(\tau)}, v \rangle_V \frac{\varphi(t) - \varphi(t+\tau)}{\tau} dt \\ &= \lim_{\tau \rightarrow 0} \int_\tau^T \langle \frac{n^{(\tau)} - \zeta_\tau n^{(\tau)}}{\tau}, v \rangle_V \varphi dt = \int_0^T \langle \chi(t), v \rangle_V \varphi(t) dt. \end{aligned}$$

Since the set $\{\varphi v \mid v \in V, \varphi \in \mathcal{C}_0^\infty(0, T)\}$ is dense in $L^q(0, T; V)$, $q = (1 + \delta)/\delta$, we have by definition of the generalized time derivative that $\partial_t n = \chi$. \square

LEMMA 11. *There exists a nonnegative function $\rho = \sqrt{n} \in L^2(0, T; V)$ such that the following convergences hold true for some subsequence of $\{\rho^{(\tau)}\}$:*

$$\begin{aligned} \rho^{(\tau)} &\rightarrow \rho \text{ in } L^{2\eta(\delta)}(0, T; W^{1,4}(\Omega)), \\ \rho^{(\tau)} &\rightarrow \rho \text{ and } \nabla \rho^{(\tau)} \rightarrow \nabla \rho \text{ a.e. in } (0, T) \times \Omega, \end{aligned}$$

for $\tau \rightarrow 0$, with the same $\delta > 0$ as in Lemma 10.

Proof. Due to the boundedness of $\{\rho^{(\tau)}\}$ in $L^2(0, T; V)$, we obtain a weakly convergent subsequence (not relabeled) and some ρ such that $\rho^{(\tau)} \rightharpoonup \rho$ in $L^2(0, T; V)$. Since $n^{(\tau)} \rightarrow n$ almost everywhere, so does $\rho^{(\tau)} = \sqrt{n^{(\tau)}} \rightarrow \sqrt{n}$ almost everywhere, which allows us to identify $\rho = \sqrt{n} \geq 0$ almost everywhere. Furthermore, we have that $\nabla \rho^{(\tau)} \rightarrow \nabla \rho$ almost everywhere. Indeed, this follows from

$$2\rho^{(\tau)} \nabla \rho^{(\tau)} = \nabla n^{(\tau)} \rightarrow \nabla n = 2\rho \nabla \rho \text{ a.e. in } (0, T) \times \Omega.$$

Hence, due to the boundedness of the sequence $\{\rho^{(\tau)}\} \subset L^{2\eta(\delta)}(0, T; W^{1,4}(\Omega))$ given by (16) and (17), along with the almost everywhere convergence of the sequence, we may apply the Lebesgue's dominated convergence in order to obtain strong convergence in the asserted space. \square

With the preceding results, we may now pass to the limit $\tau \rightarrow 0$ in (12).

Proof of Theorem 1. Let us first establish a weak convergence for $\{F^{(\tau)} \rho^{(\tau)}\}$ and $\{\Phi^{(\tau)}\}$. Due to estimate (18) we obtain $\chi \in L^2(0, T; L^2(\Omega))$ such that

$$F^{(\tau)} \rho^{(\tau)} \rightharpoonup \chi \text{ in } L^2(0, T; L^2(\Omega)).$$

for some subsequence (not relabeled). Due to the almost everywhere convergence for $\rho^{(\tau)}$ given in Lemma 11, we may define F such that $F\rho = \chi$ almost everywhere, where we set $F(t, x) = 0$ when $\rho(t, x) = 0$. As for $\{\Phi^{(\tau)}\}$, we obtain from Proposition 3 the a priori estimate $\|\Phi\|_{2,2} \leq c\|n\|_2$, which directly gives us the uniform bound for $\{\Phi^{(\tau)}\}$. Therefore, there exists some subsequence and Φ such that

$$\Phi^{(\tau)} \rightharpoonup \Phi \text{ in } L^{1+\delta}(0, T; V_0).$$

Now, by multiplying (12) with the appropriate test functions, integrating over time and space, integrating by parts, and passing to the limit $\tau \rightarrow 0$, we finally obtain

$$\begin{aligned} \langle \partial_t n, \varphi_1 \rangle &= \langle (F\rho) 2\nabla \rho, \nabla \varphi_1 \rangle + \langle (F\rho) \rho, \Delta \varphi_1 \rangle \quad \forall \varphi_1 \in L^q(0, T; V), \\ \langle F\rho, \varphi_2 \rangle &= \langle -\epsilon^2 \Delta \rho + 2\rho \log \rho - \sigma \rho \Phi, \varphi_2 \rangle \quad \forall \varphi_2 \in L^2(0, T; L^2(\Omega)), \\ \langle \Phi, \varphi_3 \rangle &= \langle n, \varphi_3 \rangle \quad \forall \varphi_3 \in H_0^1(\Omega), \end{aligned}$$

with $q = (1 + \delta)/\delta$, which completes the proof. \square

Appendix A. Without loss of generality, $\beta = 0$. Otherwise, we may simply make the shift $u' = u - \beta|\Omega| \in W_0$ and proceed with the proof for u' . From Poincaré's inequality

$$\|u - \frac{1}{|\Omega|} \int_{\Omega} u \, dx\|_p \leq c_p \|\nabla u\|_p \quad \text{for any } u \in W^{1,p},$$

we deduce that the norms $\|\cdot\|_{W_0}$ and $\|\cdot\|_{1, \frac{d}{d-1}}$ are equivalent.

We know from standard elliptic theory that a unique weak solution $u \in W_0$ of (3) exists when $f \in L^2(\Omega)$ due to the Lax-Milgram theorem, the Poincaré inequality, and the continuous embedding $H^1(\Omega) \hookrightarrow W_0$. Since $L^2(\Omega) \hookrightarrow L \log L(\Omega)$ is dense, it is sufficient to show, due to the BLT theorem, that the solution operator $S: f \mapsto u$ is bounded with respect to the norms $\|\cdot\|_{L \log L(\Omega)}$ and $\|\cdot\|_{W_0}$ respectively.

For $\phi(s) := \text{sign}(s) \log(1 + |s|)$ we set

$$\Phi(s) := (1 + |s|)(\log(1 + |s|) - 1), \quad \text{and} \quad \theta(s) := 2(1 + |s|)^{1/2}.$$

We define $\Psi(r)$ as the conjugate convex function of $\Phi(s)$, i.e.,

$$\Psi(r) := \sup_s (rs - \Phi(s)),$$

where the supremum is attained if and only if $r = \phi(s)$ [18]. We first observe that $|u| \in W^{1, \frac{d}{d-1}}(\Omega)$ for any $u \in W^{1, \frac{d}{d-1}}(\Omega)$ [17]. Moreover, we have the estimate

$$\|\nabla |u|\|_{\frac{d}{d-1}} \leq \|1 + |u|\|_{\frac{d}{d-2}}^{1/2} \|\nabla \theta(u)\|_2 \leq \frac{\delta}{2} \|u\|_{\frac{d}{d-2}} + \frac{1}{2\delta} \|\nabla \theta(u)\|_2^2 + \frac{\delta}{2} |\Omega|^{(d-2)/d},$$

for any $\delta > 0$. Clearly, this follows directly from Hölder's and Young's inequality. Due to the Sobolev embedding $W^{1, \frac{d}{d-1}}(\Omega) \hookrightarrow L^{\frac{d}{d-2}}(\Omega)$ as well as the norm equivalence $\|\cdot\|_{W_0} \sim \|\cdot\|_{1, \frac{d}{d-1}}$, we may choose an appropriate $\delta > 0$ to further obtain

$$(19) \quad \|\nabla |u|\|_{\frac{d}{d-1}} \leq \tilde{c} \|\nabla \theta(u)\|_2^2 + \frac{1}{2\tilde{c}} |\Omega|^{(d-2)/d} \quad \text{for any } u \in W_0,$$

for some constant $\tilde{c} > 0$, depending only on d and $|\Omega|$. Now, by testing equation (3) with $\phi(u)$, we obtain

$$\alpha \int_{\Omega} |\nabla \theta(u)|^2 \, dx \leq \int_{\Omega} \Phi(kf) \, dx + \int_{\Omega} \Psi\left(\frac{1}{k} \phi(u)\right) \, dx,$$

which holds for any $k > 0$. Since Φ and Ψ are convex with $\Phi(0) = \Psi(0) = 0$, we have $\Phi(ks) \leq k\Phi(s)$ and $\Psi(k^{-1}\phi(s)) \leq k^{-1}\Psi(\phi(s)) = k^{-1}(s\phi(s) - \Phi(s))$. Together with (19) we get

$$\|u\|_{W_0} \leq \frac{\tilde{c}k}{\alpha} \int_{\Omega} \Phi(f) \, dx + \frac{1}{\alpha \tilde{c}k} \int_{\Omega} ((1 + |u|) - \log(1 + |u|)) \, dx + \frac{1}{2\tilde{c}} |\Omega|^{(d-2)/d}.$$

Using the Sobolev embedding $W^{1, \frac{d}{d-1}}(\Omega) \hookrightarrow L^1(\Omega)$ and choosing $k > 0$ appropriately finally leads to the estimate

$$\|Sf\|_{W_0} = \|u\|_{W_0} \leq c \left(\|f\|_{L \log L(\Omega)} + M(|\Omega|) \right) \quad \text{for all } f \in L^2(\Omega),$$

where $c > 0$ is a constant depending only on d , $|\Omega|$ and α , and $M: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a monotonically increasing function of $|\Omega|$. Consequently, we may extend the solution operator S to a solution operator $\hat{S}: L \log L(\Omega) \rightarrow W_0$, which establishes the existence of solutions for (3). Uniqueness follows by standard arguments involving linearity and the maximum principle, and therefore concludes the proof. \square

Appendix B. Let $\sqrt{n} \in H^2(\Omega)$ and $n \in L^\infty(\Omega)$ be uniformly bounded away from zero. Then for any $\delta_1, \delta_2 > 0$ and some $\alpha \in (0, 1)$ for $d = 2$ or $\alpha \in (1/3, 1)$ for $d = 3$, there exists a constant $c > 0$ such that

$$\int_{\Omega} n^2 dx \leq \delta_1 \int_{\Omega} n \left| \frac{\nabla \sqrt{n}}{\sqrt{n}} \right|^4 dx + \delta_2 \int_{\Omega} n \left| \frac{\nabla \sqrt{n}}{\sqrt{n}} \right|^2 dx + c(\delta_1, \delta_2, \alpha, \|n\|_1),$$

where the constant c depends also on the constants from Proposition 2.

Indeed, if we set $\rho = \sqrt{n}$, then from the Proposition 2, we obtain

$$\|n\|_2^{\frac{1}{2}} = \|\rho\|_4 \leq c_1 \|\rho\|_{1,2}^{\frac{d}{4}} \|\rho\|_2^{\frac{4-d}{4}} = c_1 \|\rho\|_{1,2}^{\frac{d}{4}} \|n\|_1^{\frac{4-d}{8}}.$$

On the other hand, by setting $z = \sqrt[4]{n}$, we obtain

$$\|n\|_2^{\frac{1}{4}} = \|z\|_8 \leq c_2 \|z\|_{1,4}^{\frac{d}{8}} \|z\|_4^{\frac{8-d}{8}} = c_2 \|z\|_{1,4}^{\frac{d}{8}} \|n\|_1^{\frac{8-d}{32}}.$$

From the two inequalities above, we have

$$\|n\|_2 \leq c_1^2 \|\rho\|_{1,2}^{\frac{d}{2}} \|n\|_1^{\frac{4-d}{4}}, \quad \|n\|_2 \leq c_2^4 \|z\|_{1,4}^{\frac{d}{2}} \|n\|_1^{\frac{8-d}{8}}.$$

Now let $\alpha \in (0, 1)$ for $d = 2$, $\alpha \in (1/3, 1)$ for $d = 3$, and

$$\begin{aligned} \|n\|_2^{1-\alpha} &\leq c_1^{2(1-\alpha)} \|\rho\|_{1,2}^{\frac{(1-\alpha)d}{2}} \|n\|_1^{\frac{(1-\alpha)(4-d)}{4}}, \\ \|n\|_2^{1+\alpha} &\leq c_2^{4(1+\alpha)} \|z\|_{1,4}^{\frac{(1+\alpha)d}{2}} \|n\|_1^{\frac{(1+\alpha)(8-d)}{8}}. \end{aligned}$$

Putting them together yields,

$$\begin{aligned} \|n\|_2^2 &\leq c_1^{2(1-\alpha)} c_2^{4(1+\alpha)} \|n\|_1^{\frac{16-(3-\alpha)d}{8}} \|z\|_{1,4}^{\frac{(1+\alpha)d}{2}} \|\rho\|_{1,2}^{\frac{(1-\alpha)d}{2}} \\ &\leq \delta_1 \|z\|_{1,4}^4 + \delta_2 \|\rho\|_{1,2}^2 + \tilde{c}(\delta_1, \delta_2, \alpha, \|n\|_1) \\ &= \delta_1 \|\nabla z\|_4^4 + \delta_2 \|\nabla \rho\|_2^2 + c(\delta_1, \delta_2, \alpha, \|n\|_1) \\ &= \delta_1 \int_{\Omega} n^{-1} |\nabla \sqrt{n}|^4 dx + \delta_2 \int_{\Omega} |\nabla \sqrt{n}|^2 dx + c(\delta_1, \delta_2, \alpha, \|n\|_1), \end{aligned}$$

which is the required inequality. \square

Appendix C. In order to show the assertion, we employ the method introduced in [15] (c.f. [14]). We introduce the functions

$$\xi = \frac{|\nabla \rho|}{\rho}, \quad \eta = \frac{1}{d} \frac{\Delta \rho}{\rho}, \quad (\eta + \mu) \xi^2 = \frac{1}{\rho^3} \nabla^2 \rho : (\nabla \rho)^2,$$

where $A : (b)^2 = \sum_{i,j=1}^d a_{ij} b_i b_j$ for $A = (a_{ij}) \in \mathbb{R}^{d \times d}$, $b \in \mathbb{R}^d$, and $\varrho \geq 0$ by

$$\left| \frac{\nabla^2 \rho}{\rho} \right|^2 = d\eta^2 + \frac{d}{d-1} \mu^2 + \varrho^2,$$

which exists due to [15, Lemma 3]. Now set the functionals

$$\begin{aligned}\mathcal{J}(\rho) &= \int_{\Omega} \rho^2 \left[\left| \frac{\nabla^2 \rho}{\rho} \right|^2 + \frac{\Delta \rho}{\rho} \left| \frac{\nabla \rho}{\rho} \right|^2 - \delta \left| \frac{\nabla \rho}{\rho} \right|^4 \right] dx \\ &= \int_{\Omega} \rho^2 \left[\left(d\eta^2 + \frac{d}{d-1} \mu^2 + \varrho^2 \right) + d\eta\xi^2 - \delta\xi^4 \right] dx, \\ \mathcal{K}(\rho) &= \int_{\Omega} |\nabla^2 \rho|^2 dx = \int_{\Omega} \rho^2 \left(d\eta^2 + \frac{d}{d-1} \mu^2 + \varrho^2 \right) dx,\end{aligned}$$

and the dummy integral expression

$$\mathcal{I}(\rho) = \int_{\Omega} \operatorname{div} (\rho^{-1} |\nabla \rho|^2 \nabla \rho) dx = \int_{\Omega} \rho^2 ((d+2)\eta\xi^2 + 2\mu\xi^2 - \xi^4) dx,$$

which is zero due to the boundary condition $\partial_{\nu} \sqrt{n} = 0$ on Γ . The goal is to find constant c_0, c_1 such that $\mathcal{J} - c_0 \mathcal{K} = \mathcal{J} - c_0 \mathcal{K} + c_1 \mathcal{I} \geq 0$, and in such a way that c_0 is chosen to be as large as possible. Summing the functionals up as described, we obtain

$$(20) \quad (\mathcal{J} - c_0 \mathcal{K} + c_1 \mathcal{I})(\rho) = \int_{\Omega} \rho^2 [(1 - c_0)d\eta^2 + (d + c_1(d+2))\eta\xi^2 + Q(\mu, \xi, \varrho)] dx,$$

where Q is the polynomial in μ, ξ and ϱ given by

$$Q(\mu, \xi, \varrho) = (1 - c_0) \frac{d}{d-1} \mu^2 + 2c_1 \mu \xi^2 - (c_1 + \delta) \xi^4 + (1 - c_0) \varrho^2.$$

By choosing $c_1 = -d/(d+2)$, the second term on the right hand side of (20) vanishes. We write $Q(\mu, \xi, \varrho) = b_1 \mu^2 + 2b_2 \mu \xi^2 + b_3 \xi^4 + b_4 \varrho^2$, where

$$b_1 = \frac{(1 - c_0)d}{d-1}, \quad b_2 = -\frac{d}{d+2}, \quad b_3 = \frac{d}{d+2} - \delta, \quad b_4 = (1 - c_0),$$

and demand that $Q \geq 0$ for any given μ, ξ, ϱ . If $c_0 \leq 1$, then $b_4 \geq 0$. Now we choose $c_0 \leq 1$ in such a way that the remaining terms in Q is nonnegative as well. This is the case if $b_1 > 0$ and $b_1 b_3 - b_2^2 \geq 0$. The second condition is equivalent to

$$(1 - c_0)(d+2)(1 - (d+2)\delta/d) - (d-1) \geq 0,$$

and therefore

$$c_0 \leq 1 - \frac{d-1}{(d+2)(1 - (d+2)\delta/d)} < 1 - \frac{d-1}{d+2} = \frac{3}{d+2},$$

for δ chosen sufficiently small. Choosing such a $c_0 \in (0, 3/(d+2))$, and using again the inequality $\|\nabla^2 \rho\|_2 \leq \|\Delta \rho\|_2$, we obtain

$$(\mathcal{J} - c_0 \mathcal{K})(\rho) \geq \int_{\Omega} \rho^2 (1 - c_0) d\eta^2 dx = \frac{1 - c_0}{d} \int_{\Omega} |\Delta \rho|^2 dx \geq \frac{1 - c_0}{d} \mathcal{K}(\rho),$$

which yields further, $\mathcal{J}(\rho) \geq \gamma \mathcal{K}(\rho)$, with $\gamma = (1 + (d-1)c_0)/d$. \square

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